STOCHASTIC MODELS OF CITY POPULATION DENSITY

Robert T. Amsden, Rutgers-The State University

Previous Studies

The literature which is relevant to the problem of the spatial distributions of urban populations may be classified in two ways. These serve either to describe directly urban population densities or to examine various stochastic processes which, although not necessarily directed toward the city, nevertheless, are relevant by providing components of a quantified mechanism. The descriptive articles bring out two major points. The first is that the various cities examined seem all to have a fairly consistent pattern of population density (where density implies members per unit area). This functional form may be approximated by a negative exponential where population density is primarily a function of radius from the center. Also, this work would seem to indicate that some cities eventually reach a point in time at which the inner or central population density reaches a plateau and then proceeds to fall off as the city grows and expands further. These few articles of description do not quantify any causes of the changes in the spatial distributions of urban populations.

No articles were found in the statistical literature which were directed toward the city problem and which developed city models incorporating random processes. However, a considerable amount of theory has been developed which deals with aspects of the problem. Much effort has been devoted to stochastic birth and death processes. Recently some attention has been given to stochastic processes which involve birth and death as well as the random position of the individual. These processes have been of two types: the first being random walk models in which an individual moves forward or backward a unit step (with random waiting times as an added complexity) while undergoing birth and death. Secondly there are diffusion type processes in which the individuals continually move infinitesimal distances backward or forward while subject to birth and death probabilities.

Bailey (1968) last year commented on the availability in the literature of various stochastic processes relevant to biological populations and then went on to say, "In general, the results obtained relate to the probability distribution of the total number of individuals present in any class. The possibility of a spatial arrangement of individuals is usually ignored or not explicitly introduced as it entails a considerable increase in complexity." In this article he does bring in the spatial aspect. Populations exist at points in one, two, or three dimensional space and the individuals are subject to random birth and death as well as random translation to the nearest neighboring populations. In discussing the need for more study about the joint probabilities of his models he comments that, "Most practical applications are of course more likely to involve two or three space dimensions, but it will probably be simpler to solve the one-dimensional problems first." In his conclusion he suggests that, "Methods are therefore required not only of characterizing the behaviour of the process for general discussion, but also for testing the model against observable data."

Studies of the Author

We begin with very simple, but illuminating, non-stochastic processes involving population movement. This idealized spatial distribution of population is acted upon by a purely <u>deterministic</u> force which moves individuals in an outward direction. It is shown that this kind of process, when acting upon a distribution of population, develops a complete hole in the center and suffers from lack of any fluctuation phenomena. Lest one be misled it is impossible to construct a model of city growth and spread based upon a purely deterministic force which will not develop a hole.

In order that we may allow some individuals to remain in the center of our model city and yet require others to leave, it becomes necessary to incorporate stochastic processes. A stratified model has been developed by the author in which we study the mean or expected number of individuals in each stratum as a function of time and initial conditions; individuals may move up (or outward in the city context) one stratum at a jump after a random waiting time. This model, which is built upon the analytical methods of Seal (1945) is a new development in that, in contrast to Seal's work, we start with initial numbers of individuals in each stratum and then proceed to examine how the averages of each stratum's count changes through time. We note that this structure lacks any fluctuation phenomena and that, although we have allowed random waiting times for individuals, once they are selected they can move only one step into the next higher (or outer) stratum.

A further step has been a new application of stochastic processes theory in which random waiting times and random distances of movement in the outward direction upon selection are applied to a sectionally continuous spatial distribution model of population. Feller and others have examined stochastic processes where the element is affected by random waiting times and random distances of movement upon selection; but there do not appear to be any publications which apply these concepts to an initially spatially distributed population of individuals and then examine the time path of the population density function. The author has so done and incorporated rates of birth and death and applied some of the analytical procedures outlined by Bartlett (1966, pp. 75-78). An expected population density function is derived in which we require differential elements of individuals to move after random waiting times and then by random translations once selected. This concept is not new, but the derivation has been carried through in its entirety so that we end up with a process in which the spatial population density function changes through time by means of the movement of differential elements.

Several applications, all of which are new, were then made of this expected population density process. Some of the explicit solutions involve modified Bessel functions. One case deserves special note. The spatial distribution of population on the plane is chosen to be initially in the uniform or rectangular form, as though individuals lived evenly distributed within a walled city with no one living outside. These individuals (or differential elements) are subjected to random waiting times, random movements in the outward direction upon selection, and rates of birth and death (at time zero the walls are flattened and the outward rush begins). An explicit solution is obtained, and it is shown that, except for time zero, the population density is always slightly higher slightly outside the center. The importance of such a result is that much, although not all, of the actual city data examined in the literature evidence central holes in population density. In the publications it has been suggested that this is a result of business interests concentrated in the center. The work of this author would suggest that such an effect might arise whenever there is a tendency of the inhabitants to move in an outward direction, especially if the city population density function is relatively flat near the center whenever this outward movement begins.

Up to this point we have considered individuals to be affected by random waiting times and random distances moved in the outward direction both of which are independent of the radial position of the subject. We then set up a model in which either of these random features is related to radial position. In either case we derive a Laplace transform which satisfies a differential difference equation. Explicit solutions are not derived.

Fluctuation phenomena are then brought into the models by examining the behaviour of a spatial distribution of discrete individuals. We begin by deriving the probability distribution for position of a single individual who is subjected to random waiting times as well as random distances moved in an outward direction once selected. The probability density for being at a point is then found as a function of time with the solution involving Bessel functions. A similar result is given by Feller (1966, p. 58). We build upon this process by next considering a spread out group of individuals each of whom is subjected to the above process. The expected density of individuals at a point is derived as a function of time. This simple derivation also represents a new development, and is important because the spatial aspect is brought into the model. However, it does not evidence any fluctuation.

In order to include fluctuation phenomena resulting from random waiting times and random translations, two new processes are developed in which we derive the probability distribution of counts of individuals inside a specified interval of position. A criterion is outlined for use of the model in city planning; a unit width of annular ring is defined for the city and then one asks how far from the center one can project the model before the coefficient of variation begins to blow up. Beyond this point the model would be inapplicable; but anywhere inside the ring, the city planner could use his model for projections of future population densities. In this work we have set up general expressions for the calculation of the coefficients of variation.

We should note several points. It is a fact that the movement of people through time in a city exhibits stochastic aspects. Introduction of stochastic phenomena into city models is the only way of reflecting the birth, death, and movements taking place in city development. Stochastic processes do account for major qualitative features described by the demographers. Finally we have established a criterion for determining the stability of our stochastic models.

REFERENCES

(1) Amsden, Robert T., Stochastic Theory of Population Density, (Dissertation), Rutgers-The State University, New Brunswick, New Jersey, 1969 (See this for a more exhaustive bibliography. Copies of the dissertation may be purchased through University Microfilms, Ann Arbor, Michigan).
(2) Bailey, Norman T. J., "Stochastic Birth, Death and Migration Processes for Spatially Distributed Populations," Biometrika (1968), 55, 189-198.
(3) Bartlett, H. S., An Introduction to Stochastic Processes with Special Reference to Methods and Applications, Cambridge University Press, London, 1966.
(4) Clark, Colin, "Urban Population Densities," Journal of the Royal Statistical Society, 114, P. IV (1951), 490-496.
(5) Feller, William, An Introduction to Probability Theory and Its Applications, II, John Wiley & Sons, Inc., New York, 1966.

(6) Seal, H. L., "The Mathematics of a Population Composed of Stationary Strata Each Recruited from the Stratum Below and Supported at the Lowest Level by a Uniform Annual Number of Entrants," Biometrika 33 (1945), 226-230.